Gaussian Measures on a Banach Space

SLP Midterm Report

Varun Sunil Shah

18B090010

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Introduction

This reading project is based off of Chapter 8 of the 2^{nd} edition of 'Probability Theory - an Analytic View' by Daniel W. Stroock.

Our aim in this report is to generalize the notion of Brownian motion to any separable Banach space, and construct an abstract space that captures the same basic properties as that of Brownian motion in $\mathbb{R}^N$. This work was pioneered by Wiener and then continued by Lévy, Cameron, Martin, then the likes of Kolmogorov, Varadhan and many more. We are looking at the distribution of Brownian motion, which is called the Wiener measure, as he was the first to construct it. We construct it the same way he did: as a Gaussian measure on an infinite dimensional space. Specifically, given a Banach space E, we look at such measures which are centered Gaussian measures on E (centered as in having 0 mean), i.e., for each $x^* \in E^*, x \in E \mapsto \langle x, x^* \rangle \in \mathbb{R}$ is a Gaussian random variable with 0 mean.

1 Constructing the Classical Wiener Measure

1.1 The Classical Wiener Space

In general, we look at Brownian paths as continuous real values functions, i.e., $\mathbf{B}(t) \in C([0,\infty);\mathbb{R}^N)$. The problem with this, however, is that this space is NOT a Banach space. Hence, we need to shrink our space to make it a Banach space, and from there we can continue our investigation.

As we know that $\mathbf{B}(0) = 0$ and $\lim_{t\to\infty} t^{-1}|\mathbf{B}(t)| = 0$, we look at the space $\Theta(\mathbb{R}^N)$ of all continuous paths $\theta : [0,\infty) \to \mathbb{R}^N$ with the property that $\theta(0) = 0$ and $\lim_{t\to\infty} t^{-1}|\theta(t)| = 0$. We now arrive at the first significant lemma of our study:

Lemma 1.1 *The space $(\Theta(\mathbb{R}^N), \|.\|_{\Theta(\mathbb{R}^N)})$ is a Banach space that is continuously embedded as a Borel measurable subset of $C(\mathbb{R}^N)$, where $\|.\|_{\Theta(\mathbb{R}^N)}$ is a lower semi-continuous map given by*

$$\theta \in C(\mathbb{R}^N) \mapsto \|\theta\|_{\Theta(\mathbb{R}^N)} \equiv \sup_{t\geq 0} \frac{|\theta(t)|}{1+t} \in [0,\infty]$$

Furthermore, the dual space $\Theta(\mathbb{R}^N)^*$ can be identified with the space of $\mathbb{R}^N$-valued Borel *measures* $\boldsymbol{\lambda}$ *on* $[0,\infty)$ *such that* $\boldsymbol{\lambda}(\{0\}) = 0$ *and*

$$\|\boldsymbol{\lambda}\|_{\Theta(\mathbb{R}^N)^*} \equiv \int_{[0,\infty)} (1+t)|\boldsymbol{\lambda}|(\,dt)$$

where $|\boldsymbol{\lambda}|$ *stands for the variation measure determined by* $\boldsymbol{\lambda}$. *The duality relation for any* $\boldsymbol{\theta} \in \Theta(\mathbb{R}^N)$ *is given by*

$$\langle \boldsymbol{\theta}, \boldsymbol{\lambda} \rangle = \int_{[0,\infty)} \boldsymbol{\theta}(t) \cdot \boldsymbol{\lambda}(\,dt)$$

Lastly, if $(\boldsymbol{B}(t), \mathcal{F}_t, \mathbb{P})$ *is an* $\mathbb{R}^N$*- values Brownian motion, then* $\boldsymbol{B} \in \Theta(\mathbb{R}^N)$ $\mathbb{P}$*-almost surely and* $\mathbb{E}^{\mathbb{P}}[\|\boldsymbol{B}\|^2_{\Theta(\mathbb{R}^N)}] \leq 32N$.

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This lemma gives us a base to work off of, i.e., it shows that the space $\Theta(\mathbb{R}^N)$ is actually a separable Banach space (through lower semi-continuity of the norm), and that it contains all Brownian paths almost surely. It also gives us a convenient way to identify its dual space as a subspace of the functions of finite total variation.
In view of this fundamental lemma, we see that the distribution induced by the $\mathbb{R}^N$-valued Brownian motion gives us a Borel measure $\mathcal{W}^N$ on a separable Banach space $\Theta(\mathbb{R}^N)$. This is what we call the **classical Wiener measure**.

1.2 The Classical Wiener Measure

We will now try to characterize the classical Wiener measure, and we use the following fact about probability measures on a separable Banach space.

Lemma 1.2 *Let* E *be a separable Banach space and* E^* *denote its dual space. Then the Borel field* $\mathcal{B}_E$ *coincides with the sigma field generated by the maps* $x \in E \mapsto \langle x, x^* \rangle \in \mathbb{R}$ *for each* $x^* \in E^*$. *Furthermore, if for* $\mu \in \boldsymbol{M}_1(E)$, *we define its* ***characteristic function/Fourier transform*** $\hat{\mu} : E \to \mathbb{C}$ *by*

$$\hat{\mu}(x^*) = \int_E \exp[\sqrt{-1}\langle x, x^* \rangle]\mu(\,dx)$$

then this is a weak continuous function on* Θ^* *and if* $\mu, \nu \in \boldsymbol{M}_1(\Theta)$ *such that* $\hat{\mu} = \hat{\nu}$, *then* $\mu = \nu$.

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This lemma gives us an important tool we can use to identify the Wiener measure, as we see that as long as any measure satifies the above property, we can liken it to the classical Wiener measure and treat it the same way. We now use this lemma to find the characteristic function of $\mathcal{W}^N$. Let $(\cdot,\cdot)_V$ to denote the inner product associated with the inner-product space V.

Given an $\mathbb{R}^N$-valued Brownian motion $\mathbf{B}(t)$, note that the set $\{(\xi, \mathbf{B}(t))_{\mathbb{R}^N} : t \geq 0, \xi \in \mathbb{R}^N\}$ spans a Gaussian family in $L^2(\mathbb{P}, \mathbb{R})$. Hence, as the distributions of Brownian motion induce the Wiener measure on the set $\Theta(\mathbb{R}^N)$, we see that the set $\{(\xi, \boldsymbol{\theta}(t))_{\mathbb{R}^N} : t \geq 0, \xi \in \mathbb{R}^N\}$ spans a Gaussian family in $L^2(\mathcal{W}^N, \mathbb{R})$. Now, for any $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)$, as $\langle \boldsymbol{\theta}, \boldsymbol{\lambda} \rangle$ is an integral, it can be seen as a limit of Riemann sums, which are all centred Gaussian random variables. Thus, we see that $\boldsymbol{\theta} \mapsto \langle \boldsymbol{\theta}, \boldsymbol{\lambda} \rangle$ is a centered Gaussian random variable (limit of Gaussians is Gaussian as long as convergence is guaranteed).

Thus, the characteristic function of $\mathcal{W}^N$ is

$$\widehat{\mathcal{W}^N}(\boldsymbol{\lambda}) = \int_{\mathbb{R}^N} \exp[\sqrt{-1}\langle \boldsymbol{\theta}, \boldsymbol{\lambda} \rangle] \mathcal{W}^N(\,dx) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}^N} [\langle \boldsymbol{\theta}, \boldsymbol{\lambda} \rangle]^2 \mathcal{W}^N(\,dx)\right)$$

To show that equality, we have used the classical proof of the Gaussian characteristic function in $\mathbb{R}$, and extended it to $\mathbb{R}^N$. Now, for a Brownian motion $\mathbf{B}(t)$, as $\mathbf{B}(s+t) - \mathbf{B}(t)$ is independent of $\mathbf{B}(t)$ $\forall\, s, t \in \mathbb{R}^N$, we see that for $s, t \in \mathbb{R}^N, 0 \leq s \leq t$,

$$\begin{aligned}
\mathbb{E}^{\mathcal{W}^N}\left[(\boldsymbol{\xi}, \boldsymbol{\theta}(s))_{\mathbb{R}^N} \cdot (\boldsymbol{\eta}, \boldsymbol{\theta}(t))_{\mathbb{R}^N}\right] &= \mathbb{E}^{\mathcal{W}^N}\left[(\boldsymbol{\xi}, \boldsymbol{\theta}(s))_{\mathbb{R}^N} \cdot [(\boldsymbol{\eta}, \boldsymbol{\theta}(s))_{\mathbb{R}^N} + (\boldsymbol{\eta}, \boldsymbol{\theta}(t-s))_{\mathbb{R}^N}]\right] \\
&= \mathbb{E}^{\mathcal{W}^N}\left[(\boldsymbol{\xi}, \boldsymbol{\theta}(s))_{\mathbb{R}^N} \cdot (\boldsymbol{\eta}, \boldsymbol{\theta}(s))_{\mathbb{R}^N}\right] = s(\boldsymbol{\xi}, \boldsymbol{\eta})_{\mathbb{R}^N}
\end{aligned}$$

Thus, we use Fubini's theorem to now see that

$$\mathbb{E}^{\mathcal{W}^N}\left[\langle \boldsymbol{\theta}, \boldsymbol{\lambda} \rangle^2\right] = \iint_{[0,\infty)^2} s \wedge t \boldsymbol{\lambda}(\,ds) \boldsymbol{\lambda}(\,dt)$$

and this directly gives us

$$\widehat{\mathcal{W}^N}(\boldsymbol{\lambda}) = \exp\left(-\frac{1}{2}\iint_{[0,\infty)^2} s \wedge t \boldsymbol{\lambda}(\,ds) \boldsymbol{\lambda}(\,dt)\right)$$

Thus, we have shown that $\mathcal{W}^N$ is a centered Gaussian measure on $\Theta(\mathbb{R}^N)$ and for each $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$, the function $\boldsymbol{\theta} \mapsto \langle \boldsymbol{\theta}, \boldsymbol{\lambda} \rangle$ is a centered Gaussian random variable with variance $\iint_{[0,\infty)^2} s \wedge t \boldsymbol{\lambda}(\,ds) \boldsymbol{\lambda}(\,dt)$.

1.3 The Cameron-Martin Space

We start with a technical lemma needed for multiple theorems ahead. We will not mention the proof here.

Lemma 1.3 *Let E be a separable real Banach space and suppose that $H \subseteq E$ is a real Hilbert space that is continuously embedded as a dense subspace of E. Then:*

1. *For each $x^* \in E^*$ there is a unique $h_{x^*} \in H$ such that $(h, h_{x^*})_H = \langle h, x^* \rangle$ for all $h \in H$, and the map $x^* \mapsto h_{x^*}$ is linear, continuous, one-to-one and onto a dense subspace of H.*

2. *If $x \in E$, then $x \in H$ if and only if there is a $K < \infty$ such that $|\langle x, x^* \rangle| \leq K \|h_{x^*}\|_H \ \forall\ x^* \in E^*$. Moreover, for each $h \in H$, $\|h\|_H = \sup\{\langle h, x^* \rangle : x^* \in E^* \ \ \& \ \|x^*\|_{E^*} \leq 1\}$.*

3. *If L^* is a weak* dense subspace of E^*, then there exists a sequence $\{x_n^*, n > 0\} \subseteq L^*$ such that $\{h_{x_n^*}, n > 0\}$ is an orthonormal basis for H. Moreover, if $x \in E$, then $x \in H$ if and only if $\sum_{n=0}^{\infty} \langle x, x_n^* \rangle^2 < \infty$. Finally,*

$$(h, h')_H = \sum_{n=0}^{\infty} \langle h, x_n^* \rangle \langle h', x_n^* \rangle \ \forall\ h, h' \in H.$$

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We have shown that the Wiener measure is a centered Gaussian measure on a Banach space. However, we would love to look at it as a standard Gaussian measure on a Hilbert space (due to its much nicer properties). In finite dimensions, this is quite easy to do, as any centered Gaussian measure on $\mathbb{R}^N$ can be seen as a standard Gaussian measure on a Hilbert space H. We show this by taking a Gaussian measure X on $\mathbb{R}^N$ with mean 0 and non-degenerate covariance matrix $\mathbf{C}$. We look at H, the Hilbert space constructed by taking $\mathbb{R}^N$ with the inner product $(a, b)_H = (a, \mathbf{C}b)_{\mathbb{R}^N}$. If we take λ_H to be the standard Lebesgue measure generated on H, we see that

$$\widehat{X}(\mathbf{h}) = \exp\left(-\frac{\|\mathbf{h}\|_H^2}{2}\right)$$

Thus, X is uniquely determined to be the standard Gaussian measure on H (Lemma 1.2). However, this does not work in infinite dimensions. We may try to guess the space H where the Wiener measure might live by simply passing the limit of N to infinity in the distribution of X (á la Feynman). This gives us a naïve guess of H being the space $\mathbf{H}(\mathbb{R}^N)$ which is the space of all absolutely continuous functions $\mathbf{h} : [0, \infty) \to \mathbb{R}^N$ with $\mathbf{h}(0) = 0$ and

$$\|\mathbf{h}\|_{\mathbf{H}(\mathbb{R}^N)} = \|\dot{\mathbf{h}}\|_{L^2([0,\infty);\mathbb{R}^N)} < \infty$$

Now we show that this is indeed the required Hilbert space. Note that for any $\mathbf{h} \in \mathbf{H}(\mathbb{R}^N)$,

$$|\mathbf{h}(t)| \leq t^{\frac{1}{2}} \|\mathbf{h}\|_{\mathbf{H}(\mathbb{R}^N)} \implies \implies t^{-1}|\mathbf{h}(t)| \leq t^{-\frac{1}{2}} \|\mathbf{h}\|_{\mathbf{H}(\mathbb{R}^N)} \to 0 \text{ as } t \to \infty$$

Thus, $\mathbf{h} \in \Theta(\mathbb{R}^N)$ and $\|\mathbf{h}\|_{\Theta(\mathbb{R}^N)} \leq \frac{1}{2}\|\mathbf{h}\|_{\mathbf{H}(\mathbb{R}^N)}$. Also, as $C_c^\infty([0,\infty),\mathbb{R}^N)$ is dense in $\Theta(\mathbb{R}^N)$ and $C_c^\infty([0,\infty),\mathbb{R}^N) \subseteq \mathbf{H}(\mathbb{R}^N) \subseteq \Theta(\mathbb{R}^N)$, we have that $\mathbf{H}(\mathbb{R}^N)$ is continuously embedded as a dense subset of $\Theta(\mathbb{R}^N)$.

Thus, from Lemma 1.3, we see that $\Theta(\mathbb{R}^N)^*$ can be identified as a dense subspace of $H(\mathbb{R}^N)^* \equiv H(\mathbb{R}^N)$ and for each $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$ there is a unique $h_{\boldsymbol{\lambda}} \in H(\mathbb{R}^N)$ such that $(h, h_{\boldsymbol{\lambda}})_{H(\mathbb{R}^N)} = \langle h, \boldsymbol{\lambda}\rangle$ for all $h \in H(\mathbb{R}^N)$. In fact,

$$\begin{aligned}
\langle h, \boldsymbol{\lambda}\rangle &= \int_{[0,\infty)} h(t)\cdot\boldsymbol{\lambda}(\,dt) = \int_{[0,\infty)} \left(\int_{[0,t)} \dot{h}(\tau)\,d\tau\right)\cdot\boldsymbol{\lambda}(\,dt) \\
&= \int_{[0,\infty)} \dot{h}(\tau)\cdot\boldsymbol{\lambda}((\tau,\infty))\,d\tau, \quad \text{(Fubini's Theorem)} \\
&= (h, h_{\boldsymbol{\lambda}})_{H(\mathbb{R}^N)}, \quad h_{\boldsymbol{\lambda}}(t) = \int_{(0,t]} \boldsymbol{\lambda}((\tau,\infty))\,d\tau.
\end{aligned}$$

Now, we finally see that

$$\|h_{\boldsymbol{\lambda}}\|^2_{\mathbf{H}(\mathbb{R}^N)} = \int_{(0,\infty)} |\boldsymbol{\lambda}((\tau,\infty))|^2\,d\tau = \iint_{(0,\infty)^2} s\wedge t\boldsymbol{\lambda}(\,ds)\boldsymbol{\lambda}(\,dt)$$

Hence, we see that the characteristic function of the Wiener measure is represented by

$$\widehat{\mathcal{W}^N}(\boldsymbol{\lambda}) = \exp\left(-\frac{\|h_{\boldsymbol{\lambda}}\|^2_{\mathbf{H}(\mathbb{R}^N)}}{2}\right), \quad \boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$$

which is exactly the characteristic function of the standard Gaussian measure. Thus, on this space, they are both equivalent (using Lemma 1.2). Thus the space $\mathbf{H}(\mathbb{R}^N)$ is indeed the Hilbert space we were looking for. This space is called the **Cameron-Martin space** for the classical Wiener measure. Also, the triple $(\mathbf{H}(\mathbb{R}^N), \Theta(\mathbb{R}^N), \mathcal{W}^N)$ is called the **classical Wiener space**.

2 Abstract Wiener Spaces

2.1 The Basic Structure Theorem

We now move on to more general circumstances, i.e., we will show that given a Banach space E and a non-degenerate Gaussian measure $\mathcal{W}$ $\left(\mathbb{E}^{\mathcal{W}}[\langle x, x^*\rangle^2] = 0 \text{ if and only if } x^* = 0\right)$ on E, it has the same structure as the classical Wiener measure on $\Theta(\mathbb{R}^N)$, and thus they can be viewed equivalently. Furthermore, there is a continuously embedded Hilbert Space H in E such that $\mathcal{W}$ is the standard Gaussian measure on H. We thus call any triple $(H, E, \mathcal{W})$ which shares the same properties as the classical Wiener space an **abstract Wiener Space.**

Theorem 2.1 *Suppose that E is a separable real Banach space and that $\mathcal{W}$ is a centered Gaussian measure on E. Then there exists a unique Hilbert space H such that $(H, E, \mathcal{W})$ is an abstract Wiener space.*

The proof of this theorem is done in a roundabout way, by starting with uniqueness and then showing existence using some results proved during the uniqueness side of things. Suppose that H is a Hilbert space which satisfies the Theorem. Then $\forall\ x^*, y^* \in E^*, \langle h_{x^*}, y^* \rangle = (h_{x^*}, h_{y^*})_H = \langle h_{y^*}, x^* \rangle$. Furthermore, $\langle h_{x^*}, x^* \rangle = \|h_{x^*}\|_H^2 = \int \langle x, x^* \rangle^2 \mathcal{W}(\,dx)$.
Thus, we see that

$$\langle h_{x^*}, y^* \rangle = \int \langle x, x^* \rangle \langle x, y^* \rangle \mathcal{W}(\,dx) = \left\langle \int \langle x, x^* \rangle x \mathcal{W}(\,dx), y^* \right\rangle \tag{1}$$

Now, we use Fernique's Theorem to check that

$$\int \|\langle x, x^* \rangle x\|_E \mathcal{W}(\,dx) \leq \left(\int \|\langle x, x^* \rangle\|_E^2 \mathcal{W}(\,dx) \right)^{\frac{1}{2}} \cdot \left(\int \|x\|_E^2 \mathcal{W}(\,dx) \right)^{\frac{1}{2}} = C\|h_{x^*}\|_H^2 < \infty \tag{2}$$

Thus, we can safely see that

$$h_{x^*} = \int \langle x, x^* \rangle x \mathcal{W}(\,dx) \tag{3}$$

Given $h \in H$, pick $\{x_n^* : n \geq 1\} \subseteq E^*$ so that $h_{x_n^*} \to h$ in H. Then we get $\limsup \|\langle \cdot, x_n^* \rangle - \langle \cdot, x_m^* \rangle\|_{L^2(\mathcal{W},\mathbb{R})} = \limsup \|h_{x_n^*} - h_{x_m^*}\|_H = 0$. So if Ψ denotes the closure of $\{\langle \cdot, x^* \rangle : x^* \in E^*\}$ in $L^2(\mathcal{W}, \mathbb{R})$ and $F : \Psi \to E$ is given by

$$F(\psi) = \int x\psi(x)\mathcal{W}(\,dx), \quad \psi \in \Psi,$$

then $h = F(\psi)$ for some $\psi \in \Psi$. Conversely, if $\psi \in \Psi$ such that there are $\{x_n^* : n \geq 1\}$ and $\langle \cdot, x^* \rangle \to \psi$ in $L^2(\mathcal{W}, \mathbb{R})$, then $h_{x_n^*}$ converges to h in H and to $F(\psi)$ in E. Thus, $h = F(\psi)$. Combining both, we get that $H = F(\Psi)$ and this shows uniqueness.
For existence, we need only show that if Ψ and F are defined as above and if $H = F(\Psi)$, then $(H, E, \mathcal{W})$ is an abstract Wiener space. Now we see that

$$\langle F(\psi), x^* \rangle = \int \langle x, x^* \rangle \psi(x) \mathcal{W}(\,dx) = (F(\psi), h_{x^*})_H$$

and thus h_{x^*} has the same description as that given in (1) and (3). We also notice that $\|h_{x^*}\|_H^2$ is simply the variance of $\langle \cdot, x^* \rangle$, which is a centered Gaussian random variable. Thus, the characteristic function of $\mathcal{W}$ is given by

$$\widehat{\mathcal{W}}(x^*) = \exp\left(-\frac{1}{2}\mathbb{E}^{\mathcal{W}}[\langle x, x^*\rangle^2]\right) = \exp\left(-\frac{\|h_{x^*}\|_H^2}{2}\right)$$

Thus, $\mathcal{W}$ is the standard Gaussian measure on H. We also see that similar to (2), $\|F(\psi)\|_E \leq C\|\psi\|_{L^2(\mathcal{W},\mathbb{R})} = C\|F(\psi)\|_H$, so H is continuously embedded in E. Finally, we use the Hahn-Banach theorem and the fact that $\psi = \langle \cdot, x^* \rangle$ to show that H is dense in E. Thus, we get that H is continuously embedded as a dense subspace of E and the centered Gaussian measure $\mathcal{W}$ on E is equivalent to the standard Gaussian measure on H. This is equivalent to saying that the triple $(H, E, \mathcal{W})$ is an abstract Wiener space.

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2.2 The Cameron-Martin space and the Paley-Wiener map

Given an abstract Wiener space $(H, E, \mathcal{W})$, we call H the **Cameron-Martin space** of the abstract Wiener space. The theorem below outlines some interesting properties of H, of which we shall only prove the second part.

Theorem 2.2
1. *If $(H, E, \mathcal{W})$ is an abstract Wiener space, then the map $x^* \in E^* \mapsto h_{x^*} \in H$ is continuous from the weak* topology on E^* into the strong topology on H. In particular, $\forall\ R > 0$, $\{h_{x^*} : x^* \in \overline{B_{E^*}(0, R)}\}$ is a compact subset of H, $\overline{B_H(0, R)}$ is a compact subset of E, so $H \in \mathcal{B}_E$. Furthermore, when E is infinite dimensional, $\mathcal{W}(H) = 0$.*

2. *There is a unique linear, isometric map $\mathcal{I} : H \to L^2(\mathcal{W}, \mathbb{R})$ such that $\mathcal{I}(h_{x^*}) = \langle \cdot, x^* \rangle\ \forall\ x^* \in E^*$ and $\{\mathcal{I}(h) : h \in H\}$ is a Gaussian family in $L^2(\mathcal{W}, \mathbb{R})$.*

We define $\mathcal{I}(h_{x^*}) = \langle \cdot, x^* \rangle$. Then, for each x^*, $\mathcal{I}(h_{x^*})$ is a centered Gaussian random variable with variance $\|h_{x^*}\|_H^2$. Thus, $\mathcal{I}$ is a linear isometry from $\{h_{x^*} : x^* \in E^*\}$ into $L^2(\mathcal{W}, \mathbb{R})$. Now, as $\{h_{x^*} : x^* \in E^*\}$ is dense in H (by Lemma 1.3), we can extend the map to a linear isometry from H into $L^2(\mathcal{W}, \mathbb{R})$, and this is the required map. Moreover, as the L^2-limit of centered Gaussians is again centered Gaussian, $\mathcal{I}(h)$ is centered Gaussian $\forall\ h \in H$.

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This map $\mathcal{I}$ is called the **Paley-Wiener map**. We can look at $\{h_{x^*} : x^* \in E^*\}$ as the set of $g \in H$ such that the map $h \to (h, g)_H$ admits a continuous extension to E. Even though for infinite dimensional H, this doesn't actually work (no continuous exntension exists), we see that $\mathcal{I}(h)$ does the same job as the map above, although only upto a $\mathcal{W}$-null set. Furthermore, if we were to adopt this line of thinking, we see that

$$\mathbb{E}^{\mathcal{W}}\left[\exp(\sqrt{-1}\mathcal{I}(h))\right] = \exp\left(-\frac{\|h\|_H^2}{2}\right), \quad h \in H$$

Thus if $\mathcal{W}$ were on H, it would definitely be the standard Gaussian measure.
One final thing we note about the Paley-Wiener map is its use in the property of Gaussian measures under translation. If $y \in H$ and $\tau_y : E \to E$ such that $\tau_y(x) = x + y$, we look at the measure $(\tau_y)_*\mathcal{W}$. We posit that it will have the characteristic function

$$\widehat{(\tau_y)_*\mathcal{W}}(h) = \exp\left(-\frac{\|h-y\|_H^2}{2}\right) = \exp\left[(h, y)_H - \frac{\|y\|_H^2}{2}\right] \cdot \exp\left(-\frac{\|h\|_H^2}{2}\right)$$

Hence, if we assume that $\mathcal{I}(y)$ gives us the correct representation for $(\cdot, y)_H$, then we can guess that

$$\left[(\tau_y)_*\mathcal{W}(\,dx)\right](\,dh) = R_y(x)\mathcal{W}(\,dx), \text{ where } R_y = \exp\left[\mathcal{I}(y) - \frac{\|y\|_H^2}{2}\right]$$

The fact that the above is true was proved by Cameron and Martin, and this is why it is called the **Cameron-Martin Formula.**

2.3 From Hilbert to Abstract Wiener Spaces

Up until now, we were given a Banach space E and a centered Gaussian measure $\mathcal{W}$ and we constructed a Hilbert space H which made the triple an abstract Wiener space. Now, we go the other way around.
We will only work with real, infinite dimensional and separable spaces from now on. As we know that all real, infinite dimensional and separable Hilbert spaces are isometrically isomorphic to each other, we show that the same holds for abstract Wiener spaces.

Theorem 2.3 *Let H and H' be a pair of Hilbert spaces, and suppose that F is a linear isometry from H onto H'. Further, suppose that $(H, E, \mathcal{W})$ is an abstract Wiener space. Then there exists a separable, real Banach space $E' \supseteq H'$ and a linear isometry $\tilde{F}$ from E onto E' such that $\tilde{F} \restriction H = F$ and $(H', E', \tilde{F}_*\mathcal{W})$ is an abstract Wiener space.*

We define $\|h'\|_{E'} = \|F^{-1}h'\|_E$ for $h' \in H'$ and we let E' be the completion of H' with respect to the above norm. We see that H' is continuously embedded in E' as a dense subspace and F admits a unique extension $\tilde{F}$ as a linear isometry from E onto E'. Furthermore, if $(x')^* \in E'^*$ and $\tilde{F}^T$ is the adjoint map from $(E')^*$ onto E', then

$$(h', h'_{(x')^*})_{H'} = \langle h', (x')^* \rangle = \langle F^{-1}h', \tilde{F}^T(x')^* \rangle = (F^{-1}h', h'_{\tilde{F}^T(x')^*})_H = (h', Fh'_{\tilde{F}^T(x')^*})_{H'}$$

and thus, $h'_{(x')^*} = Fh'_{\tilde{F}^T(x')^*}$, and we get

$$\mathbb{E}^{\tilde{F}_*\mathcal{W}}\left[e^{\sqrt{-1}\langle x', (x')^* \rangle}\right] = \mathbb{E}^{\mathcal{W}}\left[e^{\sqrt{-1}\langle \tilde{F}x, (x')^* \rangle}\right] = \mathbb{E}^{\mathcal{W}}\left[e^{\sqrt{-1}\langle x, \tilde{F}^T(x')^* \rangle}\right]$$
$$= e^{\left(-\frac{1}{2}\|h_{\tilde{F}^T(x')^*}\|_H^2\right)} = e^{\left(-\frac{1}{2}\|F^{-1}h'_{(x')^*}\|_H^2\right)} = e^{\left(-\frac{1}{2}\|h'_{(x')^*}\|_H^2\right)}$$

Thus, $(H', E', \tilde{F}_*\mathcal{W})$ is an abstract Wiener space.

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Now, given any separable real Hilbert space H, let $F : \mathbf{H}(\mathbb{R}) \to H$ be an isometric isomorphism. So by the above theorem, we can see that there exists a separable Banach space E and an isometric isomorphism $\tilde{F} : \Theta(\mathbb{R}) \to E$ such that $(H, E, \tilde{F}_*\mathcal{W}^1)$ is an abstract Wiener space.

We now look at multiple ways of constructing an abstract Wiener space from a Hilbert space, one of which was adopted by Lévy in his polygonalization construction of Brownian motion. What we plan to do is choose an orthonormal basis of $\{h_n : n \geq 1\}$ of H, and as $\mathcal{W}$ is the standard Gaussian measure on H, $X_n(h) = (h, h_n)_H$ are all iid Gaussian random variables, and for each $h \in H$, the series $\sum_{n=1}^{\infty} X_n(h)h_n$ converges to h in H. So what we do is start with a sequence of iid standard normal random variables $\{X_n : n \geq 1\}$ and look at the Banach space E where the series $\sum_{n=1}^{\infty} X_n(h)h_n$ converges with probability 1 for all $h \in H$. Then we take $\mathcal{W}$ to E, and we have our abstract Wiener space.

Both Lévy and Wiener chose different orthonormal bases for their construction of Brownian motion, and this can be found in (Stroock, p. 319). Theorem 8.3.3 (Stroock, p. 320) gives us a thoeretical backdrop to perform these computations, and essentially guarantees that given the existence of Brownian motion (or in general, any Wiener measure), there are several ways to construct it.

We now show an important fact about the relation between the abstract Wiener measure $\mathcal{W}$ and the Banach space E where it resides.

Theorem 2.4 *If W is a non-degenerate, centered Gaussian measure on a separable Banach space E, then E is the support of W in the sense that W assigns positive probability to every non-empty open subset of E.*

Let H be the Cameron–Martin space for $\mathcal{W}$. Since H is dense in E, we just need to show that $\mathcal{W}(B_E(g,R)) > 0 \ \forall \ g \in H \ \& \ R > 0$. Moreover, by the Cameron–Martin formula we have $\mathcal{W}(B_E(g,R)) > C \cdot \mathcal{W}(B_E(0,R))$ for some $C > 0$. Now, we choose an orthonormal basis $\{h_n : n \geq 1\}$ and let $S_m = \sum_{n=1}^{m} X_n(h)h_n$. Then, using theorem 8.3.3, we see that

$$\mathcal{W}(B_E(0,R)) \geq \frac{1}{2}\mathcal{W}(\|S_n\|_E < \frac{R}{2}) \geq \gamma_{0,1}^{n+1}(B_{\mathbb{R}^{n+1}}(0, \frac{R}{2C_0})) > 0, \ \ \forall \ R > 0$$

where $C_0 > 0$ and $\gamma_{0,1}^{n+1}$ is the standard Gaussian measure on $\mathbb{R}^{n+1}$.

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This essentially shows us that E is the 'smallest' such Banach space from which we can construct an abstract Wiener space in the sense that every subset of E has positive $\mathcal{W}$-probability.

2.4 Orthogonality

We move our attention now to orthogonality issues, as they are extremely important to enable the construction of abstract Wiener spaces, in particular, the construction of the required Banach space E from the Hilbert space H. Given a closed linear subspace L of H, we define the projection map $\Pi_L : H \to L$ such that for each $h \in H, h - \Pi_L h \perp L$. We will see that if $(H, E, \mathcal{W})$ is an abstract Wiener space and L is a finite dimensional subspace of H, then Π_L can be almost surely extended to P_L on E, and also that $P_L x \to x$ in $L^2(\mathcal{W}, E)$ as $L \uparrow H$.
There are a few theorems mentioned in (Stroock, 2010), namely Theorems 8.3.7, 8.3.8 and 8.3.9 regarding the various properties of the projection maps and its relation to the Paley-Wiener map. I will write down the theorems, but will not prove them.

Theorem 2.5 (8.3.7) *Let $(H, E, \mathcal{W})$ be an abstract Wiener space and $\{h_n : n \geq 0\}$ be an orthonormal basis in H. Then, for each $h \in H, \sum_{m=0}^{\infty}(h, h_m)_H \mathcal{I}(h_m)$ converges to $\mathcal{I}(h)$ almost surely and in $L^p(\mathcal{W}, \mathbb{R})$ for every $p \in [1, \infty)$.*

Theorem 2.6 (8.3.8) *Let $(H, E, \mathcal{W})$ be an abstract Wiener space. For each finite dimensional subspace L of H there is an almost surely unique map $P_L : H \to L$ such that for every $h \in H$ and almost surely every $x \in E, (h, P_L x)_H = \mathcal{I}(\Pi_L h)(x)$. Further, if $\dim(L) = k$ and $\{g_1, \ldots, g_k\}$ is an orthonormal basis for L, then $P_L x = \sum_{i=1}^{k} [\mathcal{I}(g_i)](x) g_i$, and thus $P_L x \in L$ for almost every x in E. Finally, $X \to P_L x$ is $\mathcal{W}-$independent of $x \to x - P_L x$.*

Theorem 2.7 (8.3.9) *Let $(H, E, \mathcal{W})$ be an abstract Wiener space and $\{h_n : n \geq 0\}$ be an orthonormal basis for H. Set $L_n = span(\{h_0, h_1, \dots, h_n\})$. Then, for all $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\mathbb{E}^{\mathcal{W}}[\|P_L x\|_E^2] \leq \epsilon^2$ whenever L is a finite dimensional subspace that is perpendicular to L_n.*

An important thing to note is that this construction of E is by no means unique, in fact there are uncountably many such E which give rise to an abstract Wiener space. We show this in the following theorem:

Theorem 2.8 *If $(H, E, \mathcal{W})$ is an abstract Wiener space, then there exists a separable Banach space E_0 that is continuously embedded in E as a measurable subset and has the properties that $\mathcal{W}(E_0) = 1$, bounded subsets of E_0 are relatively compact in E, and $(H, E_0, \mathcal{W} \upharpoonright E_0)$ is again an abstract Wiener space.*

I will show the outline of the proof here, and the details can be found in (Stroock, 2010, p 324).
We choose $\{x_n^*; n \geq 0\} \subseteq E^*$ so that $\{h_{x_n^*} : n \geq 0\}$ is an orthonormal basis for H. We set $L_n = \text{span}(\{h_{x_0^*}, h_{x_1^*}, \dots, h_{x_n^*}\})$. Using Theorem 2.7, we choose an increasing sequence $\{n_m\}_{m\geq 0}$ so that $n_0 = 0$ and $\mathbb{E}^{\mathcal{W}}[\|P_L x\|_E^2]^{\frac{1}{2}} \leq 2^{-m}$ for $m \geq 1$ and finite dimensional $L \perp L_{n_m}$. We now define Q_l on E into H such that

$$Q_0 x = \langle x, x_0^* \rangle h_{x_0^*} \text{ and } Q_l x = \sum_{n=n_{l-1}+1}^{n_l} \langle x, x_n^* \rangle h_{x_n^*} \text{ when } l \geq 1$$

Finally, we take $S_m = \sum_{l=0}^{m} Q_l$, and define E_0 to be the set of $x \in E$ such that

$$\|x\|_{E_0} \equiv \|Q_0 x\|_E + \sum_{l=1}^{\infty} l^2 \|Q_l x\|_E < \infty \text{ and } \|S_m x - x\|_E \to 0.$$

The rest of the proof shows that $\|\cdot\|_{E_0}$ is a norm on E_0 and this gives us a Banach space. It goes on to show that the triple $(H, E_0, \mathcal{W} \upharpoonright E_0)$ is an abstract Wiener space. Thus, for any Hilbert space, there are infinitely many abstract Wiener spaces possible, depending on the choice of E.

■

We now move on to the concept of orthogonal invariance and, in particular, the extension of teh orthogonal invariance property of Gaussian measures on a finite Banach space to the infinite dimensional case. We know that the standard Gaussian measure on $\mathbb{R}^N$ is invariant to any rotational transformation. In particular, if $\mathcal{O}$ is any orthogonal matrix, then the Gaussian measure is invariant under the map $T_{\mathcal{O}} : \mathbb{R}^N \to \mathbb{R}^N$ such that $T_{\mathcal{O}}(x) = \mathcal{O}x$. We now look at the analogue when $\mathbb{R}^N$ is replaced by an abstract Wiener space $(H, E, \mathcal{W})$. We see the following result:

Theorem 2.9 *Let $(H, E, \mathcal{W})$ be an abstract Wiener space and $\mathcal{O}$ an orthogonal transformation on H. Then there is a $\mathcal{W}$-almost surely unique, Borel measurable map $T_{\mathcal{O}} : E \to E$ such that $\mathcal{I}(h) \circ T_{\mathcal{O}} = \mathcal{I}(\mathcal{O}^T h)$ almost surely for each $h \in H$. Moreover, $\mathcal{W} = (T_{\mathcal{O}})_* \mathcal{W}$.*

To prove existence, we choose an orthonormal basis $\{h_n : n \geq 0\}$ for H and let C be the set of $x \in E$ for which both $\sum_{m=0}^{\infty} [\mathcal{I}(h_m)](x) h_m$ and $\sum_{m=0}^{\infty} [\mathcal{I}(h_m)](x) \mathcal{O} h_m$ converge in E. Now, we use Theorem 8.3.3 of (Stroock, 2010) to show that $\mathcal{W}(C) = 1$ and that

$$x \to T_{\mathcal{O}} x \equiv \begin{cases} \sum_{m=0}^{\infty} [\mathcal{I}(h_m)](x) \mathcal{O} h_m & x \in C \\ 0 & x \notin C \end{cases}$$

has a distribution $\mathcal{W}$. Hence, we have proved that $T_{\mathcal{O}}$ exists almost surely. We just need to show that $\mathcal{I}(h) \circ T_{\mathcal{O}} = \mathcal{I}(\mathcal{O}^T h)$ almost surely for each $h \in H$. We see that

$$[\mathcal{I}(h_{x^*})](T_{\mathcal{O}} x) = \langle T_{\mathcal{O}} x, x^* \rangle = \sum_{n=0}^{\infty} (h_{x^*}, \mathcal{O} h_m)_H [\mathcal{I}(h_m)](x) = \sum_{n=0}^{\infty} (\mathcal{O}^T h_{x^*}, h_m)_H [\mathcal{I}(h_m)](x)$$

which, by Theorem 2.5, converges almost surely to $\mathcal{I}(\mathcal{O}^T h_{x^*})$. Thus, we have that $\mathcal{I}(h) \circ T_{\mathcal{O}} = \mathcal{I}(\mathcal{O}^T h)$ almost surely. To show that this is true for all $h \in H$, we see that $h \mapsto \mathcal{I}(h) \circ T_{\mathcal{O}}$ and $h \mapsto \mathcal{I}(\mathcal{O}^T h)$ are both isometric maps and are equal on $\{h_{x^*} : x^* \in E^*\}$, which is dense in H. Thus they are also equal on H, and we have proved existence.
For uniqueness, note that if T and T' are two maps satisfying the conditions. then for each $x^* \in E^*$,

$$\langle Tx, x^* \rangle = \mathcal{I}(h_{x^*})(Tx) = \mathcal{I}(\mathcal{O}^T h_{x^*})(x) = \mathcal{I}(h_{x^*})(T'x) = \langle T'x, x^* \rangle$$

for almost every $x \in E$. Now, since E^* is weak* separable, we have that for almost every $x \in E$, $Tx = T'x$.

■

We now talk about ergodicity properties of abstract Wiener measures under orthogonal transformations. In finite dimensions, as all radial processes are invariant under $T_{\mathcal{O}}$ for every $\mathcal{O}$, we see that the orthogonal transformation of the Gaussian measure cannot be ergodic. However, this is not true in infinite dimensions. We specifically claim that $T_{\mathcal{O}}$ cannot be ergodic if $\mathcal{O}$ has a non-trivial finite dimensional invariant subspace L, since then we would have an orthonormal basis $\{h_1, h_2, \ldots, h_n\}$ of L, and the function $\sum_{m=1}^{n} \mathcal{I}(h_m)^2$ is non-constant and $T_{\mathcal{O}}$-invariant. Thus we arrive at the following theorem, which we state but do not prove.

Theorem 2.10 *Let $(H, E, \mathcal{W})$ be an abstract Wiener space. If $\mathcal{O}$ is an orthogonal transformation on H with the property that, for every $g, h \in H$, $\lim_{n\to\infty}(\mathcal{O}^n g, h)_H = 0$, then $T_{\mathcal{O}}$ is ergodic.*

■

This just provides us with a better picture of the Gaussian measure on infinite dimensional spaces. We move on to generalize some popular finite dimensional properties of Gaussian measures.

3 Large Deviations and Strassen's Law

3.1 Large Deviation Theory

We now generalize the notion of large deviations and the law of iterated logarithms to any abstract Wiener space. In particular, we see the following result:

Theorem 3.1 *Let $(H, E, \mathcal{W})$ be an abstract Wiener space, and for $\epsilon > 0$, let $\mathcal{W}_\epsilon$ denote the distribution obtained by transforming $\mathcal{W}$ by the map $x \mapsto \epsilon^{\frac{1}{2}} x$. Then, for any $\Gamma \in \mathcal{B}_E$,*

$$-\inf_{h\in\mathring{\Gamma}} \frac{\|h\|_H^2}{2} \le \liminf_{\epsilon\to 0} \epsilon \log \mathcal{W}_\epsilon(\Gamma) \le \limsup_{\epsilon\to 0} \epsilon \log \mathcal{W}_\epsilon(\Gamma) \le -\inf_{h\in\overline{\Gamma}} \frac{\|h\|_H^2}{2}$$

We will only show the lower bound proof here, and the proof for the upper bound, which is significantly more involved, can be found in (Stroock, 2010, p 338).
For the lower bound, all we need to show is that for any $h \in H$ and $r > 0$, we have

$$\liminf_{\epsilon\to 0} \epsilon \log \mathcal{W}_\epsilon(B_E(h, r)) \ge -\frac{\|h\|_H^2}{2} \tag{4}$$

Note that for any $x^* \in E^*$ and $\delta > 0$, using the Cameron-Martin formula,

$$\mathcal{W}_\epsilon(B_E(h_{x^*}, \delta)) = \mathcal{W}(B_E(\epsilon^{-\frac{1}{2}} h_{x^*}, \epsilon^{-\frac{1}{2}}\delta)) \ge e^{-\delta\epsilon^{-1}\|x^*\|_{E^*} - \frac{1}{2\epsilon}\|h_{x^*}\|_H^2} \mathcal{W}(B_E(0, \epsilon^{-\frac{1}{2}}\delta))$$

Now, as $\{h_{x^*} : x^* \in E^*\}$ is dense in H, we have that for all $h \in H$ and $r > 0$, there exists a $\delta > 0$ and h_{x^*} such that

$$B_E(h,r) \supseteq B_E(h_{x^*},\delta) \text{ and } \liminf_{\epsilon\to 0} \epsilon \log \mathcal{W}_\epsilon(B_E(h_{x^*},\delta)) \geq -\delta\|x^*\|_{E^*} - \frac{\|h\|_H^2}{2}$$

Thus, by taking $\delta \to 0$, we get that (1) holds, and by extension, the lower bound of Theorem 3.1 holds as well.

■

Using this, we move to state a generalized version of the Law of Iterated Logarithms, enabling us to get a sharper bound on asymptotic convergences in an abstract Wiener space. This was first proved by Strassen for Brownian motion, and hence is called Strassen's Law of Iterated Logarithms (this is technically an extension of Strassen's law, but the etymology remains).

Theorem 3.2 (Strassen's Law) *Suppose that $\mathcal{W}$ is a non-degenerate, centered, Gaussian measure on the Banach space E, and let H be its Cameron-Martin space. Let $\{X_n : n \geq 1\}$ be a sequence of independent, E-valued, $\mathcal{W}$-distributed random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\Lambda_n = \sqrt{2n\log_{(2)}(n \vee 3)}$ and $\tilde{S_n} = \frac{1}{\Lambda_n}\sum_{k=1}^n X_k$. Then, $\mathbb{P}$-almost surely, the sequence $\{\tilde{S_n} : n \geq 1\}$ is relatively compact in E and the closed unit ball $\overline{B_H(0,1)}$ in H coincides with its set of limit points. Equivalently, $\mathbb{P}$-almost surely, $\lim_{n\to\infty} \|\tilde{S_n} - \overline{B_H(0,1)}\|_E = 0$ and, for each $h \in \overline{B_H(0,1)}$, $\liminf_{n\to\infty} \|\tilde{S_n} - h\|_E = 0$.*

■

This law has tremendous applications, even in the next section, to show convergence of some unknown processes to the distributions of known ones, and to describe the magnitude of fluctuations of the convergence.

4 Euclidean Free Fields

4.1 The Ornstein-Uhlenbeck Process

As we have built up some theory on abstract Wiener spaces, we now look at applications of the same in the construction of quantum free fields (free in the sense that they don't interact). Our first example is that when the parameter set is one-dimensional and the process is basically a variation of Brownian motion. It is called the **Ornstein-Uhlenbeck**

Process.
Let $x \in \mathbb{R}^N$ and $\theta \in \Theta(\mathbb{R}^N)$. Consider the equation

$$\mathbf{U}(t,x,\theta) = x + \theta(t) + \frac{1}{2}\int_0^t \mathbf{U}(\tau,x,\theta)\,d\tau, \quad t \geq 0$$

We can use Gronwall's inequality to show that this equation has atmost one solution for a fixed x and θ. Now, if we take $\mathbf{U}(t,0,\theta) = e^{-\frac{t}{2}}\int_0^t e^{\frac{\tau}{2}}\,d\boldsymbol{\theta}(\tau)$, then by integrating by parts, we get a unique solution to be

$$\mathbf{U}(t,x,\theta) = e^{-\frac{t}{2}}x + \mathbf{U}(t,0,\theta)$$

This stochastic process $\{\mathbf{U}(t,x) : t \geq 0\}$ under $\mathcal{W}^N$ in particular, i.e., $\mathbf{U}(t,x,\theta)$ where θ is a centered Gaussain random variable, is called the Ornstein-Uhlenbeck process. We can look at this process as a Brownian motion that has a restoring force applied to it that slowly pushes it to the origin. So, locally, it looks like a Brownian motion, but goes to the origin as $t \to \infty$.
Now, we can show that the span of $\{(\xi, \mathbf{U}(t,0))_{\mathbb{R}^N} : t \geq 0 \;\&\; \xi \in \mathbb{R}^N\}$ is a Gaussian family in $L^2(\mathcal{W}^N, \mathbb{R})$, and

$$\mathrm{Cov}\left[\mathbf{U}(s,0), \mathbf{U}(t,0)\right] = \left(e^{-\frac{|t-s|}{2}} - e^{-\frac{t+s}{2}}\right)\mathbf{I}_n$$

Thus, as Gaussian measures are characterized by their mean and covariance, we see that this process has the same distribution as $\{e^{-\frac{t}{2}}\mathbf{B}(e^t - 1) : t \geq 0\}$, where $\{\mathbf{B}(t) : t \geq 0\}$ is a Brownian motion. Furthermore, we see that $\mathbf{U}(\cdot, x)$ has the distribution $\gamma_{e^{-\frac{t}{2}}x,(1-e^{-t})\mathbf{I}_n}$, so as $t \to \infty$, we see that this distribution moves to the standard Gaussian on $\mathbb{R}^N$.
This motivates us to look at the process $\{\mathbf{U}_A(t) : t \geq 0\}$ under $\gamma_{0,\mathbf{I}_n} \times \mathcal{W}^N$, i.e., $\mathbf{U}(t,x,\theta)$ where x is a standard normal random variable and θ is a centered Gaussain random variable, which we call the **ancient Ornstein-Uhlenbeck process.**
If $\{\mathbf{U}_A(t) : t \geq 0\}$ is an ancient Ornstein-Uhlenbeck process, then we see that the span of $\{(\xi, \mathbf{U}_A(t))_{\mathbb{R}^N} : t \geq 0 \;\&\; \xi \in \mathbb{R}^N\}$ is a Gaussian family with covariance

$$\mathrm{Cov}\left[\mathbf{U}_A(s), \mathbf{U}_A(t)\right] = \left(e^{-\frac{|t-s|}{2}}\right)\mathbf{I}_n$$

So, if $\{\mathbf{B}(t) : t \geq 0\}$ is a Brownian motion, then $\{e^{-\frac{t}{2}}\mathbf{B}(e^t) : t \geq 0\}$ is an ancient Ornstein-Uhlenbeck process. Further, this process is also stationary and time-reversible.
We take this motivation to define another version of this process called the **reversible Ornstein-Uhlenbeck process.** We define $\mathbf{U}_R : [0,\infty) \times \mathbb{R}^N \times \Theta(\mathbb{R}^N)^2 \to \mathbb{R}^N$ by

$$\mathbf{U}_R(t,x,\theta_+,\theta_-) = \begin{cases} \mathbf{U}_R(t,x,\theta_+) & t \geq 0 \\ \mathbf{U}_R(-t,x,\theta_-) & t < 0 \end{cases}$$

and we consider the process $\mathbf{U}_R(\cdot, x, \theta_+, \theta_-)$ under $\gamma_{0,\mathbf{I}_n} \times \mathcal{W}^N \times \mathcal{W}^N$. This process also spans a Gaussian family and has the same covariance as the ancient Ornstein-Uhlenbeck process, but now it is for all $s, t \in \mathbb{R}$, not just for positive s, t.

We can also view this process another way. By starting with a Brownian motion $\{\mathbf{B}(t) : t \geq 0\}$, we get a reversible Ornstien-Uhlenbeck process by looking at $\{e^{-\frac{t}{2}}\mathbf{B}(e^t) : t \in \mathbb{R}\}$.

Now that we are familiar with the Ornstien-Uhlenbeck process (I will refer to it as the OU process from now on), we start to characterize it formally.

4.2 The OU Process as an abstract Wiener process

We aim to show that both the one-sided and the reversible OU processes are indeed abstract Wiener processes. We start with the one-sided OU process $\{\mathbf{U}(t, 0) : t \geq 0\}$ which can be written as $\{e^{-\frac{t}{2}}\mathbf{B}(e^t - 1) : t \geq 0\}$. Thus we should look at the Hilbert space $H^U(\mathbb{R}^N)$ of functions $h^U(t) = e^{-\frac{t}{2}}h(e^t - 1)$, where $h \in H(\mathbb{R}^N)$ is the Cameron-Martin space for the classical Wiener measure. Thus, we define a map $F^U : H(\mathbb{R}^N) \to H^U(\mathbb{R}^N)$ and define a norm $\|\cdot\|_{H^U}$ that makes F^U an isometry. We get

$$\begin{aligned}\|h^U\|^2_{H^U(\mathbb{R}^N)} &= \int_{[0,\infty)} \left[\frac{d}{ds}\Big((1+s)^{\frac{1}{2}}h^U(\log(1+s))\Big)\right]^2 ds \\ &= \|\dot{h}^U\|^2_{L^2([0,\infty),\mathbb{R}^N)} + (\dot{h}^U, h^U)_{L^2([0,\infty),\mathbb{R}^N)} + \frac{1}{4}\|h^U\|^2_{L^2([0,\infty),\mathbb{R}^N)}\end{aligned}$$

Now, we can see that $(\dot{h}^U, h^U)_{L^2([0,\infty),\mathbb{R}^N)} = 0$, (we use the fact that $\lim_{t\to\infty}|h^U(t)|$ is equivalent to $\lim_{t\to\infty} t^{-\frac{1}{2}}|h(t)|$, which goes to 0 for $h \in H(\mathbb{R}^N)$). Thus,

$$\|h^U\|_{H^U(\mathbb{R}^N)} = \sqrt{\|\dot{h}^U\|^2_{L^2([0,\infty),\mathbb{R}^N)} + \frac{1}{4}\|h^U\|^2_{L^2([0,\infty),\mathbb{R}^N)}}$$

The usual proof now completes H^U with respect to this norm, but we know that the OU process lives on $\Theta^U(\mathbb{R}, \mathbb{R}^N)$, the space of $\theta \in \Theta(\mathbb{R}^N)$ such that $\lim_{t\to\infty}(\log t)^{-1}|\theta(t)| = 0$ with Banach norm $\|\theta\| \equiv \sup_{t\geq 0}(\log(e+t))^{-1}|\theta(t)|$, so we take that as our Banach space. We now state the following fact. The proof of this can be found in (Stroock, 2010, p 347).

Thus, we arrive at the following theorem, which gives us formal infrastructure to work on the OU process, and to define the quantum field where this resides.

Theorem 4.1 *Let $\mathcal{U}^N$ be the distribution of $\{\boldsymbol{U}(t, 0) : t \geq 0\}$ under $\mathcal{W}^N$. Then, the triple $(H^U(\mathbb{R}^N), \Theta^U(\mathbb{R}, \mathbb{R}^N), \mathcal{U}^N)$ is an abstract Wiener space. Further, let $H^1(\mathbb{R}, \mathbb{R}^N)$ be the space* *of absolutely continuous h satisfying*

$$\|h\|_{H^1(\mathbb{R},\mathbb{R}^N)} = \sqrt{\|\dot{h}\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} + \frac{1}{4}\|h\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}}$$

Let $\mathcal{U}_R^N$ *be the distribution of* $\{\boldsymbol{U}_R(t) : t \in \mathbb{R}\}$ *under* $\gamma_{0,\boldsymbol{I}_n} \times \mathcal{W}^N \times \mathcal{W}^N$. *Then, the triple* $(H^1(\mathbb{R},\mathbb{R}^N), \Theta^U(\mathbb{R},\mathbb{R}^N), \mathcal{U}_R^N)$ *is an abstract Wiener space.*

■

This was an example to show the wide applications of this thoery. We now move to the case when the parameter set is not one-dimensional, and attempt to construct higher dimension free fields.

4.3 Higher Dimension Free Fields

We now want to look at an analogue of the OU process when $N = 1$ and the parameter set has dimension $\nu > 1$. We will need very nice functions for this, as just L^2-ness will not work. Thus, we turn to the Schwartz function space, which is the space of all functions who's derivative is rapidly decreasing. We will show that this choice actually works. Our idea is to complete the Schwartz function space with respect to a particular norm to create a Banach space where our distribution will live. We start by looking at the space $H^1(\mathbb{R}^\nu, \mathbb{R})$ which is the space obtained by completing the Schwartz function space $\mathcal{S}(\mathbb{R}^\nu, \mathbb{R})$ with respect to the norm

$$\|h\|_{H^1(\mathbb{R}^\nu,\mathbb{R})} = \sqrt{\|\nabla h\|^2_{L^2(\mathbb{R}^\nu,\mathbb{R})} + \frac{1}{4}\|h\|^2_{L^2(\mathbb{R}^\nu,\mathbb{R})}}$$

But there is a problem with this. When $\nu \geq 2$, we see that there are some elements of $H^1(\mathbb{R}^\nu, \mathbb{R})$ that are not even defined pointwise, much less be continuous. Thus, to construct an abstract Wiener space, we have to look at Banach spaces which have generalized functions. We approach it as follows:

Define **Bessel's operator** B^s on $\mathcal{S}(\mathbb{R}^\nu, \mathbb{C})$ such that the Fourier transform of B^s is

$$\widehat{B^s\phi}(\xi) = (\frac{1}{4} + |\xi|^2)^{-\frac{s}{2}}\widehat{\phi}(\xi)$$

In particular, we see that $\|\phi\|_{H^1(\mathbb{R}^\nu,\mathbb{R})} = \|B^{-1}\phi\|_{L^2(\mathbb{R}^\nu,\mathbb{R})}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^\nu, \mathbb{R})$.

We now move to define the space $H^s(\mathbb{R}^\nu, \mathbb{R})$ to be the separable Hilbert space obtained by completing $\mathcal{S}(\mathbb{R}^\nu, \mathbb{C})$ with respect to the following norm

$$\|h\|_{H^s(\mathbb{R}^\nu,\mathbb{R})} = \|B^{-s}h\|_{L^2(\mathbb{R}^\nu,\mathbb{R})} = \sqrt{\frac{1}{(2\pi)^\nu}\int_{\mathbb{R}^\nu}(\frac{1}{4} + |\xi|^2)^s|\hat{h}(\xi)|^2\, d\xi}$$

When $s = 0$, this space is just $L^2(\mathbb{R}^\nu, \mathbb{R})$. In general, for $s \in \mathbb{R}$, we see that $H^s(\mathbb{R}^\nu, \mathbb{R}) \subseteq [\mathcal{S}(\mathbb{R}^\nu, \mathbb{R})]^*$. In fact, we can also see that $H^s(\mathbb{R}^\nu, \mathbb{R})$ is the isometric image of $L^2(\mathbb{R}^\nu, \mathbb{R})$ under the map B^s, i.e., $H^s(\mathbb{R}^\nu, \mathbb{R}) = B^s(L^2(\mathbb{R}^\nu, \mathbb{R}))$.In general, $H^{s_2}(\mathbb{R}^\nu, \mathbb{R}) = B^{s_2 - s_1}(H^{s_1}(\mathbb{R}^\nu, \mathbb{R}))$. Thus we have an isometric isomorphism between any two spaces H^{s_1} and H^{s_2} and therefore, if we are able to construct an abstract Wiener space for any $s \in \mathbb{R}$, we can construct them all as long as we know how Bessel's operator acts on the space.

We end the section and the report with two theorems on the construction of an abstract Wiener space which are pretty involved, hence the proofs will not be mentioned here. They can be found in (Stroock, 2010, p 350) and (Stroock, 2010, p 352).

Theorem 4.2 *The space $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})$ is continuously embedded as a dense subspace of the separable Banach space $C_0(\mathbb{R}^\nu, \mathbb{R})$ whose elements are continuous functions that tend to 0 at infinity and whose norm is the uniform norm. Moreover, given a totally finite, signed Borel measure λ on $\mathbb{R}^\nu$, the function*

$$h_\lambda(x) \equiv \frac{\pi^{\frac{1-\nu}{2}}}{\Gamma(\frac{\nu+1}{2})} \int_{\mathbb{R}^\nu} e^{-\frac{|x-y|}{2}} \lambda\,(dy)$$

is an element of $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})$, and

$$\|h_\lambda(x)\|_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})} = \frac{\pi^{\frac{1-\nu}{2}}}{\Gamma(\frac{\nu+1}{2})} \iint_{\mathbb{R}^\nu \times \mathbb{R}^\nu} e^{-\frac{|x-y|}{2}} \lambda\,(dx) \lambda\,(dy)$$

and

$$\langle h, \lambda \rangle = (h, h_\lambda)_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})} \quad \text{for each } h \in H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})$$

■

The next theorem allows us to extend our study of the OU process to parameter sets of more than one dimension, and we will see analogous results here.

Theorem 4.3 *Let $\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})$ be the space of continuous $\theta : \mathbb{R}^\nu \to \mathbb{R}$ satisying* $\lim_{|x| \to \infty} (\log(e + |x|))^{-1} |\theta(x)| = 0$ *and construct a separable Banach space from this using the norm* $\|\theta\|_{\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})} = \sup_{x \in \mathbb{R}^N} (\log(e + |x|))^{-1} |\theta(x)|$. *Then, $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})$ is contiunuously embedded as a dense subspace of $\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})$ and there is a $\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})}$ such that the triple*

$$(H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R}), \Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R}), \mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu, \mathbb{R})})$$

is an abstract Wiener space. Moreover, for each $\alpha \in (0, \frac{1}{2})$*, almost every* θ *is Hölder continuous of order* α*, and for each* $\alpha > \frac{1}{2}$*, almost no theta is anywhere Hölder continuous of order* α*.*

■

Thus, we are able to construct higher dimension free fields of any dimension ν using Bessel's operator.
Abstract Wiener spaces are very powerful, and we have only seen limited applications here. Further study is to be done to study the consequences and applications of this idea of an abstract Wiener space.

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Varun Sunil Shah

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Introduction

This reading project is based off of Chapter 10 of the 2^{nd} edition of 'Probability Theory - an Analytic View' by Daniel W. Stroock.

Our aim in this report is to build a bridge between partial differential equations and Brownian motion. This will provide a computational tool to the probabalists in the sense that the usual Wiener integrals reduce to solving a partial differential equation, while it provides a different representation of classical PDE solutions which give rise to more nuanced properties, We start with martingales and their connection with typical PDE's, and then move on to extending martingale representations, and then give the Arcsine law. Then we go on to recurrence, transience and the Markov property of the Wiener measure and end with some other heat kernels that result from the above properties.

1 Martingales and Partial Differential Equations

The connection between Brownian motion and PDE's originates from the fact that the Gauss kernel

$$g^{(N)}(t,x) = (2\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}},\ (t,x) \in (0,\infty) \times \mathbb{R}^N$$

is simultaneously the density for the normal distribution $\gamma_{0,tI}$ and the solution to the heat equation $\partial_t u = \frac{1}{2}\Delta u$ in $(0,\infty) \times \mathbb{R}$ with initial condition δ_0. Moreover, if $\phi \in C^b(\mathbb{R}^N, \mathbb{R})$, then

$$u_\phi(t,x) = \int_{\mathbb{R}^N} g^{(N)}(t, y-x)\phi(y)dy$$

is the only bounded $u \in C^{1,2}((0,\infty) \times \mathbb{R}^N, \mathbb{R})$ that solves that Cauchy IVP $\partial_t u = \frac{1}{2}\Delta u$ in $(0,\infty) \times \mathbb{R}^N$ with $\lim_{t\to 0} u(t,\cdot) = \phi$ uniformly on compacts. Proving that u is a solution is trivial, but uniqueness isn't. But if one assumes a little more about u, i.e., that $u \in C_b^{1,2}([0,\infty) \times \mathbb{R}^N, \mathbb{C})$, then theorem 7.16 of Stroock, 2010 tells us that when $(\mathbf{B}(t), \mathcal{F}_t, \mathbb{P})$ is a Brownian motion, for each $T > 0$, $(u(T - T \wedge t, x + \mathbf{B}(t \wedge T)), \mathcal{F}_t, \mathbb{P})$ is a martingale. Thus,

$$u(T,x) = \mathbb{E}[\phi(\mathbf{B}(T))] = \int_{\mathbb{R}^N} \phi(x+y)\gamma_{0,tI}(dy) = u_\phi(T,x)$$

). Ahead, we state a refinement of Theorem 7.1.6 that will enable us to remove the assumption that the derivatives of u are bounded.

As we see, probability theory offers us a way to lift questions about PDE's to the pathspace setting, and martingales are the perfect vehicale to do so. We now move on to some interesting properties about them.

1.1 Localizing and Extending Martingale Representations

We will now try to obtain a quite general way to represent solutions to PDE's as a Wiener integral. The proofs for all these theorems can be found in Stroock, 2010. From now on, we will look at $\mathcal{W}^N$ as a Borel measure on the Polish space $C(\mathbb{R}^N)$, because we want to consider all translates $\mathcal{W}_x^N$ of $\mathcal{W}^N$, i.ie. all distributions of $\psi \mapsto x + \psi$ under $\mathcal{W}^N$. As the translation map is continuous, the resulting function is still Borel measurable.

Theorem 1.1 *Let $\mathcal{G}$ be a non-empty open subset of $\mathbb{R} \times \mathbb{R}^N$, and, for $s \in \mathbb{R}$, define $\zeta_s^{\mathcal{G}} : C(\mathbb{R}^N) \to [0, \infty)$ by*

$$\zeta_s^{\mathcal{G}}(\psi) = inf\{t \geq 0 : (s + t, \psi(t)) \notin \mathcal{G}\}$$

Further, suppose that $V : \mathcal{G} \to \mathbb{R}$ is a Borel measurable function that is bounded above on the whole of $\mathcal{G}$ and bounded below on each compact subset of $\mathcal{G}$, and set

$$E_s^V(t, \psi) = exp\left(\int_0^{t \wedge \zeta_s^{\mathcal{G}}} V(s + \tau, \psi(\tau))d\tau\right)$$

If $w \in C^{1,2}(\mathcal{G}, \mathbb{R}) \cap C_b(\overline{\mathcal{G}}, \mathbb{R})$ satisfies $(\partial_t + \frac{1}{2}\Delta + V)w \geq f$ on $\mathcal{G}$, where $f : \mathcal{G} \to \mathbb{R}$ is a bounded Borel measurable function, then

$$\left(E_s^V(t, \psi)w(s + t \wedge \zeta_s^{\mathcal{G}}(\psi), \psi(t \wedge \zeta_s^{\mathcal{G}})) - \int_0^{t \wedge \zeta_s^{\mathcal{G}}(\psi)} E^V(\tau, \psi)f(S + \tau, \psi(\tau)), \mathcal{F}_t, \mathcal{W}_x^N\right)$$

is a submartingale for every $(s, x) \in \mathcal{G}$. In particular, if $(\partial_t + \frac{1}{2}\Delta + V)w = f$ on $\mathcal{G}$, then the above triple is a martingale.

■

The most important corollary of this is the famous **Feynman-Kac formula.**

Lemma 1.2 *Let $V : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be a Borel measurable function that is uniformly bounded above everywhere and bounded below uniformly on compacts. If $u \in C^{1,2}((0, T) \times \mathbb{R}^N, \mathbb{R})$ is bounded and satisfies the Cauchy IVP $\partial_t u = \frac{1}{2}\delta u + Vu + f$ in $(0, T) \times \mathbb{R}^N$ with $\lim_{t \to 0} u(t, \cdot) = \phi$ uniformly on compacts for some bounded, borel measurable $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ and $\phi \in C_b(\mathbb{R}^N, \mathbb{R})$, then*

$$u(T, x) = \mathbb{E}^{\mathcal{W}_x^N}\left[e^{\int_0^T V(\tau, \psi(\tau))d\tau}\phi(\psi(T))\right] + \mathbb{E}^{\mathcal{W}_x^N}\left[\int_0^T e^{\int_0^T V(\tau, \psi(\tau))d\tau} f(t, \omega(t))dt\right]$$

■

1.2 Minimum Principles

We now mention the weak and strong maximum principles in the context of the Wiener measure. The proof is quite elegant and is mentioned in Stroock, 2010.

Theorem 1.3 (Weak Maximum Principle) *Let $\mathcal{G}$ be a non-empty open subset of $\mathbb{R} \times \mathbb{R}^N$, and let V be a function of the sort described above. Further, suppose that $(s,x) \in \mathcal{G}$ is a point at which*

$$\mathcal{W}_x^N\left(\exists\, t \in (0,\infty) : (s-t, \psi(t)) \notin \mathcal{G}\right) = 1$$

If $u \in C^{1,2}(\mathcal{G},\mathbb{R})$ is bounded below and satisfies $\partial_t u - \frac{1}{2}\Delta u - Vu \geq 0$ in $\mathcal{G}$ and if, for every $(t_0, y_0) \in \partial\mathcal{G}$, $\liminf_{(t,y)\to(t_0,y_0)} u(t,y) \geq 0$, with $t_0 < s$, then $u(s,x) \geq 0$. ■

Theorem 1.4 (Strong Maximum Principle) *Let $\mathcal{G}$ be a non-empty open subset of $\mathbb{R} \times \mathbb{R}^N$, and let V be a function of the sort described above. Further, suppose that $(s,x) \in \mathcal{G}$ is a point at which*

$$\mathcal{W}_x^N\left(\exists\, t \in (0,\infty) : (s-t, \psi(t)) \notin \mathcal{G}\right) = 1$$

If $u \in C^{1,2}(\mathcal{G},\mathbb{R})$ is bounded below and satisfies $\partial_t u - \frac{1}{2}\Delta u - Vu \geq 0$ in $\mathcal{G}$ and if, for every $(t_0, y_0) \in \partial\mathcal{G}$, $\liminf_{(t,y)\to(t_0,y_0)} u(t,y) \geq 0$, with $t_0 < s$, then $u(s,x) \geq 0$. ■

The proof of Theorem 1.3 is a direct application of Theorem 1.1 and Fatou's Lemma. While Theorem 1.4 is stronger and seems like it completely takes over from the Weak Maximum principle, the weak maximum principle has more applicability due to its weaker pre-requisites. In particular, the weak maximum principle does not require that the function actually attains its minimum, while the strong maximum principle does. We have shown that the Wiener measure is a centered Gaussian measure on a Banach

1.3 The Arcsine Law

Now for particular functions, there are very few V's for which explicit solutions can be written down for PDE's of the form $\partial_t u = \frac{1}{2}\Delta u + Vu$. However, if V is independent of t, and when $N = 1$, then there exists a closed form solution. We show this by taking the Laplace Transform U_λ of u, and the PDE reduces to the ODE

$$(\lambda - \frac{1}{2}\Delta - V)U_\lambda = f$$

Thus, we can use the Feynman-Kac formula to find a closed form solution for U_λ and then u. This reasoning is what prompted the derivation of the Arcsine law.

Theorem 1.5 (Arcsine Law) *For every $T \in (0,\infty)$ and $\alpha \in [0,1]$,*

$$\mathcal{W}^1\left(\left\{\psi \in C(\mathbb{R} : \frac{1}{T}\int_0^T \mathbf{1}_{[0,\infty)}(\psi(t))dt \le \alpha\right\}\right) = \frac{2}{\pi}\arcsin(\sqrt{\alpha})$$

The proof of this theorem is done by taking a double Laplace transform of the distribution function

$$F(\alpha) = \mathcal{W}^1\left(\left\{\psi \in C(\mathbb{R} : \frac{1}{T}\int_0^T \mathbf{1}_{[0,\infty)}(\psi(t))dt \le \alpha\right\}\right)$$

■

An important corollary of the above, using Donsker's Invariance Principle, is as follows.

Lemma 1.6 *If $\{X_n : n \ge 1\}$ is a sequence of independent, uniformly square integrable random variables with mean value 0 and variance 1 on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then, $\forall\ \alpha \in [0,1]$,*

$$\lim_{n\to\infty}\left(\left\{\omega : \frac{N_n(\omega)}{n} \le \alpha\right\}\right) = \frac{2}{\pi}\arcsin(\sqrt{\alpha})$$

where $N_n(\omega)$ is the number of $m \in \mathbb{Z}^+ \cap [0,n]$ for which $S_m(\omega) \equiv \sum_{l=1}^m X_l(\omega)$ is non-negative.

The reason this law is well-known is due to the startling result that follows from it - that given a fixed $\delta \in (0, \frac{1}{2})$, the α which minimizes $\lim_{n\to\infty} \mathbb{P}(\frac{N_n}{n} \in (\alpha - \delta, \alpha + \delta) mod 1)$ is not 0.5, as people normally think (due to the Law of Large Numbers), but it is equally likely to be either 0 or 1. There are many other interesting applications of this in other fields as well.

1.4 Recurrence and Transience of Brownian Motion

We now see results about the recurrence and transience of Brownian motion derived through PDE's.

Theorem 1.7 *For $r \in [0,\infty)$, define $\zeta_r(\psi) = \inf\{t \in [0,\infty) : |\psi(t)| = r\}, \quad \psi \in C(\mathbb{R}^N)$. Then, for $|x| < r$,*

$$\mathbb{E}^{\mathcal{W}_x^N}[\zeta_r] = \frac{r^2 - |x|^2}{N}$$
$$\mathbb{E}^{\mathcal{W}_x^N}[\zeta_r^2] = \frac{(N+4)r^2 - N|x|^2}{N^2(N+2)}(r^2 - |x|^2)$$

In addition, if $0 < r < |x| < R < \infty$, *then*

$$\mathcal{W}_x^N(\zeta_r < \zeta_R) = \begin{cases} \frac{R-|x|}{R-r} & N = 1 \\ \frac{\log(R)-\log(|x|)}{\log(R)-\log(r)} & N = 2 \\ \left(\frac{r}{|x|}\right)^{N-2} \frac{R^{N-2}-|x|^{N-2}}{R^{N-2}-r^{N-2}} & N \geq 3 \end{cases}$$

In particular,

$$\mathcal{W}_x^1(\zeta_0 < \infty) = 1 \quad \forall\, x \in \mathbb{R}$$
$$\mathcal{W}_x^2(\zeta_0 < \infty) = 0, x \neq 0 \quad but \quad \mathcal{W}_x^2(\zeta_r < \infty) = 1, \quad x \in \mathbb{R}^2,\ r > 0$$
$$\mathcal{W}_x^N(\zeta_r < \infty) = \left(\frac{r}{|x|}\right)^{N-2} \quad 0 < r < |x|, \quad N \geq 3$$

Finally, we see that if $N \geq 3$,

$$\mathcal{W}_x^N\left(\lim_{t\to\infty} |\psi(t)| = \infty\right) = 1, \quad x \in \mathbb{R}^N$$

Thus, we see that in 1 and 2 dimensions, Brownian motion is always recurrent, however, in all dimensions greater than 3, it is transient.

2 The Markov Property and Potential Theory

2.1 The Markov Property for Wiener Measure

We now give the Markov property for the Wiener measure and use it to connect Brownian motion to potential theory and PDE's.

We need some notation first. Denote the time-shift map by $\Sigma_t : C(\mathbb{R}^N) \to C(\mathbb{R}^N)$, where $\Sigma_t(\psi(\tau)) = \psi(t+\tau),\ \ \tau \in [0,\infty)$. Further, if ζ is a stopping time, then $\Sigma_\zeta : \{\psi : \zeta(\psi) < \infty\} \to C(\mathbb{R}^N)$ is given by $\Sigma_\zeta(\psi(t)) = \psi(\zeta(\psi) + t)$.

Theorem 2.1 *If ζ is a stopping time and $F : C(\mathbb{R}^N) \times C(\mathbb{R}^N) \to [0,\infty)$ is a $\mathcal{F}_\zeta \times \mathcal{F}_{C(\mathbb{R}^N)}$ function, then*

$$\int_{\{\psi:\zeta(\psi)<\infty\}} F(\psi, \Sigma_\zeta(\psi))\mathcal{W}_x^N(d\psi) = \int_{\{\psi:\zeta(\psi)<\infty\}} \left(\int_{C(\mathbb{R}^N)} F(\psi,\psi')\mathcal{W}_{\psi(\zeta)}^N(d\psi')\right)\mathcal{W}_x^N(d\psi)$$

Theorem 2.1 is essentially the Markov property for the Wiener measure. More precisely, it is the Strong Markov Property for the Wiener measure.

2.2 The Dirichlet Problem

We now move to probably the most successful application of probability theory to PDE's. We start with some notation and introduction.
Let G be a non-empty, open, connected subset of $\mathbb{R}^N$. Given an $f \in C_b(G,\mathbb{R})$, we say that $u \in C^2(G,\mathbb{R})$ solves the Dirichlet problem for f in G if $\Delta u = 0$ in G and for each $a \in \partial G,\ u(x) \to f(a)$ if $x \in G \to a$. The weak maximum principle says that there is atmost one solution to the Dirichlet problem for a fixed $f \in C_b(G,\mathbb{R})$.
We call a function u harmonic on a set G if $u \in C^2(G,\mathbb{R})$ and $\Delta u = 0$. Also, if μ is a non-zero finite measure on E and $f : E \to \mathbb{R}$ is μ-integrable, then we denote

$$⨍ f d\mu = \frac{1}{\mu(E)} \int f d\mu$$

Finally, $\zeta_G : C(\mathbb{R}^N) \to [0,\infty]$, given by $\zeta_G(\psi) = \inf\{t \geq 0 : \psi(t) \notin G\}$ denotes the first exit time from G.

Theorem 2.2 *Let G be a non-empty subset of $\mathbb{R}^N$. If $u \in C_b(\overline{G},\mathbb{R})$, and $u \restriction G$ is harmonic, and $x \in G$ such that $\mathcal{W}_x^N(\zeta^G < \infty) = 1$, then*

$$u(x) = \mathbb{E}^{\mathcal{W}_x^N}[u(\psi(\zeta^G)), \zeta^G < \infty]. \tag{1}$$

In particular, if u is harmonic on G, then

$$\overline{B(x,r)} \text{ is a compact subset of } G \implies u(x) = ⨍_{\mathbb{S}^{N-1}} u(x + r\omega)\lambda_{\mathbb{S}^{N-1}}(d\omega). \tag{2}$$

Conversely, if $u : G \to \mathbb{R}$ is a locally bounded Borel measurable function that satisfies (2), then $u \in C^\infty(G,\mathbb{R})$ and u is harmonic. Finally, if $\partial G \to \mathbb{R}$ is a bounded, Borel measurable function then the function $u : G \to \mathbb{R}$ given by (1) is a bounded harmonic function on G.

Thus, if the above holds for all $x \in G$, and we wish to solve the Dirichlet problem for a fixed f, we just have to show that u given by (1) above is a solution. As we already know that this is harmonic, all that remains is to do is find conditions when it satisfies the boundary conditions of the Dirichlet problem.
We will show that if f is continuous at $a \in \partial G$ and if

$$\lim_{\substack{x \to a \\ x \in G}} \mathcal{W}_x^N(\zeta^G \geq \delta) = 0 \ \forall \ \delta > 0,$$

then the function u tends to $f(a)$ as $x \to a$ through G. Thus, we call a point $a \in \partial G$, a regular point if the above holds, and we denote it by $a \in \partial_{reg} G$.
Let $\zeta_s^G = \inf\{t \geq s : \psi(t) \notin G\}$ be the first exit time from G after time s. We now have the following lemma:

Lemma 2.3 *Regularity is a local property in the sense that, for each* $r \in (0, \infty), a \in \partial_{reg} G$ *if and only if* $a \in \partial_{reg}(G \cap B(a, r))$. *Furthermore,* $a \in \partial_{reg} G \iff a \in \partial G$ *and* $\mathcal{W}_a^N([\lim_{s \to 0} \zeta_s^G] > 0) = 0$, *which implies that* $\partial_{reg} G$ *is Borel measurable. Finally, if* $a \in \partial_{reg} G$, *then for each* $\delta > 0$,

$$\lim_{\substack{x \to a \\ x \in G}} \mathcal{W}_x^N((\zeta^G, \psi(\zeta^G)) \in (0, \delta) \times B(a, \delta)) = 1.$$

Now, directly from Theorem 2.2 and Lemma 2.3, we get the following theorem which classifies the boundary conditions for the Dirichlet problem in terms of regular points.

Theorem 2.4 *Let* G *be a non-empty open subset of* $\mathbb{R}^N$ *and* $f : \partial G \to \mathbb{R}$ *be a bounded, Borel measurable function. If* u *is given by Theorem 2.2, then* u *is a bounded harmonic function in* G, *and, for every* $a \in \partial_{reg} G$ *at which* f *is continuous,* $u(x) \to f(a)$ *as* $x \to a$ *through* G.

Thus, we have completely classified the solution to the Dirichlet problem in terms of probability theory and have given a closed form explicit solution along with the boundary conditions.

3 Other Heat Kernels

Our whole motivation for connecting probability theory and PDE's was that the heat kernel had the same form as the Gaussian distribution, or more appropriately, if $\phi \in C_b(\mathbb{R}^N, \mathbb{R})$, then the unique solution to the heat equation that tends to ϕ as $t \to 0$ is

$$u(t,x) = \int_{\mathbb{R}^N} \phi(y) g^N(t, y-x) dy$$

where $g^N(t, y-x)$ is the probability of a Brownian path going from x to y in time t.
We now look at some other functions that are both fundamental solutions to the heat equation, and the density of a Brownian motion transitioning at the same time, under different conditions.

3.1 A General Construction

For each $t > 0$, let $E_t : C(\mathbb{R}^N, \mathbb{R}) \to [0, \infty)$ be a $\mathcal{F}_t$-measurable function such that

$$E_{s+t}(\psi) = E_s(\psi) E_t(\Sigma_s(\psi)), \quad s, t \in (0, \infty), \psi \in C(\mathbb{R}^N)$$

and define

$$q(t,x,y) = \mathbb{E}^{\mathcal{W}^N}\left[E_t(x(1-l_t) + \theta_t + y l_t)\right] g^N(t, y-x), \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$$

where $l_t(\tau) = \frac{\tau \wedge t}{t}$, $\tau \in [0, \infty)$ and $\theta_t = \theta - \theta(t) l_t$, $\theta \in \Theta(\mathbb{R}^N)$. The following theorem characterizes this function.

Theorem 3.1 *For each $t \in (0, \infty)$ and Borel measurable $\phi : \mathbb{R}^N \to \mathbb{R}$ that is bounded below,*

$$\int_{\mathbb{R}^N} \phi(y) q(t,x,y) dy = \mathbb{E}^{\mathcal{W}_x^N}\left[E_t(\psi)\phi(\psi(t))\right].$$

Moreover, for all $s, t \in (0, \infty)$, and $x, y \in \mathbb{R}^N$, q satisfies the Chapman-Kolmogorov Equation,

$$q(s+t, x, y) = \int_{\mathbb{R}^N} q(s,x,z) q(t,z,y) dz.$$

Finally, if, for each $t > 0$, E_t is reversible in the sense that

$$E_t(\psi) = E_t(\tilde{\psi}^t), \quad \psi \in C(\mathbb{R}^N),$$

where $\tilde{\psi}^t(\tau) = \psi(t - t \wedge \tau), \tau \in [0, \infty)$, then $q(t,x,y) = q(t,y,x)$ for all $(t,x,y) \in (0,\infty) \times (\mathbb{R}^N)^2$.

Thus, we see that $q(t,x,y)$ could be proposed as an alternate heat kernel. We now look at a particular example, i.e., the Dirichlet Heat Kernel.

3.2 The Dirichlet Heat Kernel

Let G be a non-empty open subset of $\mathbb{R}^N$, and set $E_t^G(\psi) = \mathbf{1}_{(t,\infty)}(\zeta^G(\psi))$. We can see that this function satisfies all the necessary properties. Denoting the corresponding $q(t,x,y)$ by $p^G(t,x,y)$, we see that $p^G(t,x,y) = 0$ unless $x, y \in G$. Further, we see by the above theorem that

$$\begin{gathered}\int_G \phi(y)p^G(t,x,y)dy = \mathbb{E}^{\mathcal{W}_x^N}\left[\phi(\psi(t)), \zeta^G(\psi) > t\right], \quad (t,x) \in (0,\infty)\times G,\\ p^G(s+t,x,y) = \int_G p^G(s,x,z)p^G(t,z,y)dz, \quad (s,x),(t,y) \in (0,\infty)\times G,\\ p^G(t,x,y) = p^G(t,y,x) \quad (t,x,y) \in (0,\infty)\times G^2\end{gathered}$$

Further work is required to show that it is smooth, and is given in Stroock, 2010.
The reason we call it the Dirichlet Heat Kernel is the following theorem.

Theorem 3.2 *For each $\phi \in C_b(G,\mathbb{R})$, the function $u(t,x) = \mathbb{E}^{\mathcal{W}_x^N}\left[\phi(\psi(t)), \zeta^G(\psi) > t\right]$ is a smooth solution to the boundary value problem*

$$\begin{aligned}&\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) && \text{in } (0,\infty)\times G,\\ &\lim_{t\to 0} u(t,\cdot) = \phi && \text{uniformly on compacts,}\\ &\lim_{\substack{(t,x)\to(s,a)\\ x\in G}} u(t,x) = 0 && (s,a) \in (0,\infty)\times \partial_{reg}G.\end{aligned}$$

Moreover, if $\partial G = \partial_{reg}G$, then u is the only bounded solution to the BVP.

3.3 Feynman-Kac Heat Kernels

Let $V : \mathbb{R}^N \to \mathbb{R}$ be a Borel measurable function that is bounded above, and define

$$q^V(t,x,y) = \mathbb{E}^{\mathcal{W}^N}\left[exp\left(\int_0^t V(x+\theta_t+(y-x)l_t(\tau)d\tau\right)\right] g^N(t,y-x).$$

By letting $E_t(\psi) \equiv exp\left(\int_0^t V(\psi(\tau))d\tau\right)$, we see that all the required properties in Theorem 3.1 are satisfied. We further see that if $u \in C_b^{1,2}((0,\infty)\times\mathbb{R}^N,\mathbb{R})$ saatisfies the Cauchy IVP

$$\partial_t u = \frac{1}{2}\Delta u + Vu \quad \text{with } \lim_{t\to 0} u(t,\cdot) = \phi \text{ uniformly on compacts.}$$

for some $\phi \in C_b(\mathbb{R}^N, \mathbb{R})$, then

$$u(t,x) = \int_{\mathbb{R}^N} \phi(y) q^V(t,x,y) dy \quad (t,x) \in (0,\infty) \times \mathbb{R}^N.$$

Under some more suitable conditions, we can show that the RHS of the above equation is necessarily the solution of the IVP, and for that reason, q^V is called the **Feynman-Kac Heat Kernel with potential V**.

References

Stroock D. W. (2010). *Probability Theory. An Analytic View.* 2nd ed. Cambridge University Press.